Entanglement Theory - Part 2

Mixed state entanglement is more complicated... (& unsolved)

entanglement entropy clearly isn't a good measure in this case! $I/4$ & $|\beta^+\rangle$ have same entropy!

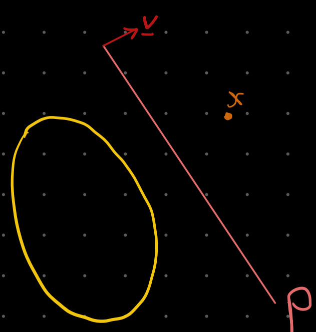
Entanglement Witnesses

The set of Bipartite separable quantum states

$$D_{sep} = \left\{ \rho_{AB} \text{ s.t. } \rho_{AB} = \sum_k \underbrace{p_k}_{\text{prob. dist.}} \sigma_{k_A} \otimes \tau_{k_B} \right\}$$

form a convex set & is contained in the larger set of all bipartite quantum states D.

> ### Hahn-Banach Corollary
>
> Given a closed convex set C and a point x outside C there exists a hyperplane P that seperates x from C.



Recall, a hyperplane P is defined by some vector $\underline{v}$ orthogonal to the plane

s.t $P = \{ x : \langle x, v \rangle = 0 \}$

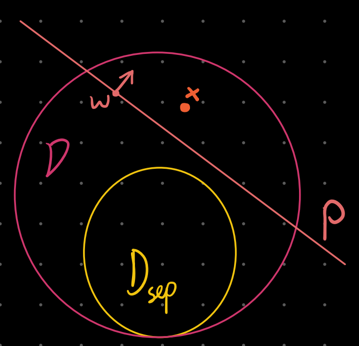
One side of P will be the points $\{ x : \langle x, v \rangle > 0 \}$ & the other $\{ x : \langle x, v \rangle < 0 \}$.

As D_{sep} is itself a convex set the Hahn-Banach corollary applies to it!

The relevant vector space = all states & observables on AB sit in space of all $d^2 \times d^2$ Hermitian matrices

Inner product = Hilbert Schmidt inner product $Tr(X^\dagger Y)$

From H-B, given an entangled state ρ_{AB} there exists a hyperplane P in the space of Hermitian operators that separates ρ_{AB} from the space of separable states.



This hyperplane is defined by some Hermitian operator W which we call an entanglement witness

An entanglement witness W is a Hermitian observable such that $Tr(W \sigma_{AB}) \geq 0 \quad \forall \quad \sigma_{AB} \in D_{sep}$

$\therefore$ if we can find such a W (& we can find such a W for any entangled state) if we find $Tr(\rho_{AB} W) \leq 0$ we know that ρ_{AB} is entangled!

But, more generally, if we are told that W is an entanglement witness & find $Tr(W\rho_{AB}) > 0$ ρ_{AB} could be entangled or unentangled.

The Peres-Horodecki Criterion

Summarising pgs. 26-38 in 5 mins...

Core idea - Remember the Choi-Jamiolkowski representation

Any channel can be associated with the positive operator

$$J(\phi) := \phi \otimes I \left(|vec(I)\rangle\langle vec(I)| \right)$$

Well the converse is also true... can associate positive operators with a channel...

$$\phi(X) = Tr_2\left((I \otimes X^T) J \right)$$

We can use this equivalence to convert the statement that we can always find a positive operator W s.t. if $Tr(W\rho_{AB}) \leq 0$ then ρ_{AB} is entangled into a statement being able to find +ve super-operators.

but not necessarily completely positive!

> A state σ_{AB} is entangled iff there exists a positive superoperator Φ such that $(\Phi \otimes I)\,\sigma_{AB}$ has a negative eigenvalue.

(for proof see typed notes)

If Φ is completely +ve then by definition $(\Phi \otimes I)\,\sigma_{AB}$ has only +ve eigenvalues....

So Φ will need to be +ve but not completely +ve

Key example of such a channel, as we saw before, is the transpose....

becomes iff for 2-qubit state

A quantum state ρ_{AB} is entangled if $(T \otimes I)\,\rho_{AB}$ has a negative eigenvalue.

We can define the 'negativity' of a quantum state as a 'measure' of mixed state entanglement

$$N(\rho_{AB}) = \frac{1}{2} \sum_{k} |\lambda_k| \quad \text{s.t. } \lambda_k < 0$$

sum of absolute value of the negative eigenvalues of ρ_{AB}.

$\boxed{\begin{array}{l}\text{For \underline{qubits}, we have that every positive superoperator } \Phi \\ \text{can be written as} \\ \qquad \Phi(X) = \mathcal{E}_1(X) + \mathcal{E}_2(X^T) \\ \text{where } \mathcal{E}_1 \text{ and } \mathcal{E}_2 \text{ are completely +ve maps \& } X^T \text{ is the} \\ \text{transpose of } X\end{array}}$